Episode 16

Describing Motion of Rigid Bodies Part 1

ENGN0040: Dynamics and Vibrations

Allan Bower, Yue Qi

School of Engineering
Brown University

Topics for todays class

Describing motion of rigid bodies

1. Review of some linear algebra
2. Describing rotations (2D/3D rotation matrices)
3. Angular velocity and angular acceleration
4. Relation between rotation matrix and angular velocity (spin matrix)
5. Formulas for the relative velocity/acceleration of two points in a rigid body
6. Proofs and derivations (optional!)

Review: some linear algebra operations

Definition: 2x2 matrix $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ 3x3 matrix $\mathbf{A}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Identity matrix: $\mathbf{1}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{1}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Transpose (switch rows and columns): $\mathbf{A}^T=\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$ $\mathbf{A}^T=\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

Matrix-vector product: $\mathbf{Ax}=\mathbf{b}\equiv\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} a_{11}x_1+a_{12}x_2 \\ a_{21}x_1+a_{22}x_2 \end{bmatrix}$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} a_{11}x_1+a_{12}x_2+a_{13}x_3 \\ a_{21}x_1+a_{22}x_2+a_{23}x_3 \\ a_{31}x_1+a_{32}x_2+a_{33}x_3 \end{bmatrix}$

Note $\mathbf{1x}=\mathbf{x}$

Dyadic product of two vectors:

$$\mathbf{a}\otimes\mathbf{b}=\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\otimes\begin{bmatrix} b_1 & b_2 \end{bmatrix}=\begin{bmatrix} a_1b_1 & a_1b_2 \\ a_2b_1 & a_2b_2 \end{bmatrix}$$

$$\mathbf{a}\otimes\mathbf{b}=\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\otimes\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}=\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{bmatrix}$$

Review: some linear algebra operations

Adding matrices: $$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

$$\mathbf{A}+\mathbf{B}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}+\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

Matrix products: $$\mathbf{AB}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

$$\mathbf{AB}=\begin{bmatrix} \boxed{a_{11} \quad a_{12} \quad a_{13}} \\ a_{21} \quad a_{22} \quad a_{23} \\ a_{31} \quad a_{32} \quad a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}=\begin{bmatrix} \boxed{a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

Note $\mathbf{AB} \neq \mathbf{BA}$ $\quad \mathbf{1A}=\mathbf{A}$ $\quad \mathbf{A1}=\mathbf{A}$

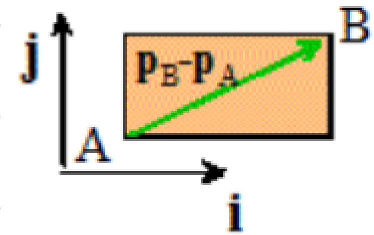
Orthogonal matrix (transpose is equal to inverse): $\mathbf{R}^T\mathbf{R}=\mathbf{R}\mathbf{R}^T=\mathbf{1}$

6 RIGID BODY DYNAMICS

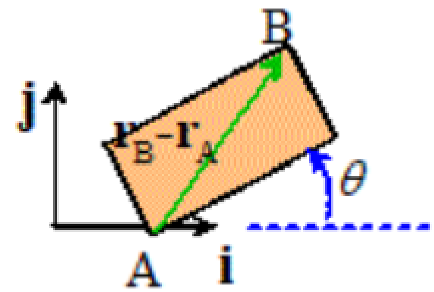
Rigid Body: an object with fixed shape capable of translation & rotation

6.1 Describing motion of rigid bodies

6.1.1 2D rotations



Rotations can be represented as a matrix (tensor) that maps vectors from initial to final orientation



$$\underline{r}_B - \underline{r}_A = R(\underline{p}_B - \underline{p}_A)$$

In 2D $$R = \begin{bmatrix} R_{xx} & R_{xy} \\ R_{yx} & R_{yy} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Observation

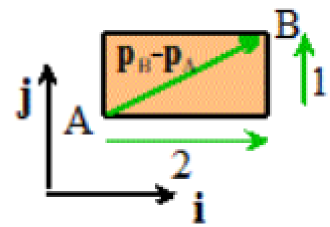
R has property that $R^T R = R R^T = \mathbb{1}$ (identity)

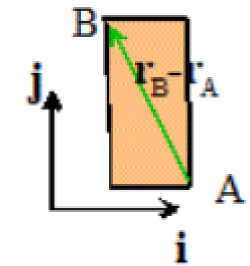
eg $$R R^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

R is "orthogonal" : a length preserving mapping

6.1.2 Example: Find the rotation tensor for the rotation shown. Hence, find the vector $\mathbf{r}_B - \mathbf{r}_A$





Note $\theta = \pi/2$

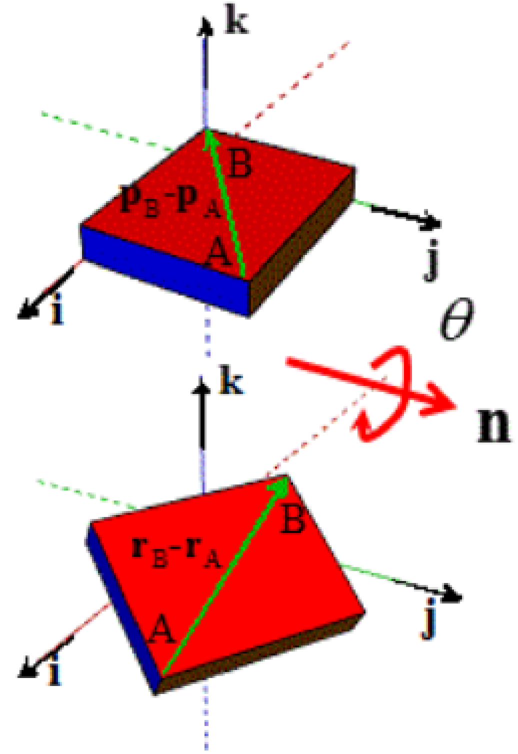
Formula $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad \underline{p}_B - \underline{p}_A = 2\underline{i} + \underline{j}$

$$\underline{r}_B - \underline{r}_A = R(\underline{p}_B - \underline{p}_A) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow \boxed{\underline{r}_B - \underline{r}_A = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\underline{i} + 2\underline{j}}$$

6.1.3 3D Rotations

3D rotations can be represented as a rotation through angle θ about an axis parallel to unit vector $\underline{n}$ (use RH screw convention)



Let $\underline{n} = n_x \underline{i} + n_y \underline{j} + n_z \underline{k}$

Then

$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

Inverse relation : Let $R = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix}$

Then

$$(1+2\cos\theta) = R_{xx} + R_{yy} + R_{zz}$$

$$\underline{n} = \frac{1}{2\sin\theta}\left\{ (R_{zy}-R_{yz})\,\underline{i} + (R_{xz}-R_{zx})\,\underline{j} + (R_{yx}-R_{xy})\,\underline{k} \right\}$$

To see this check R matrix

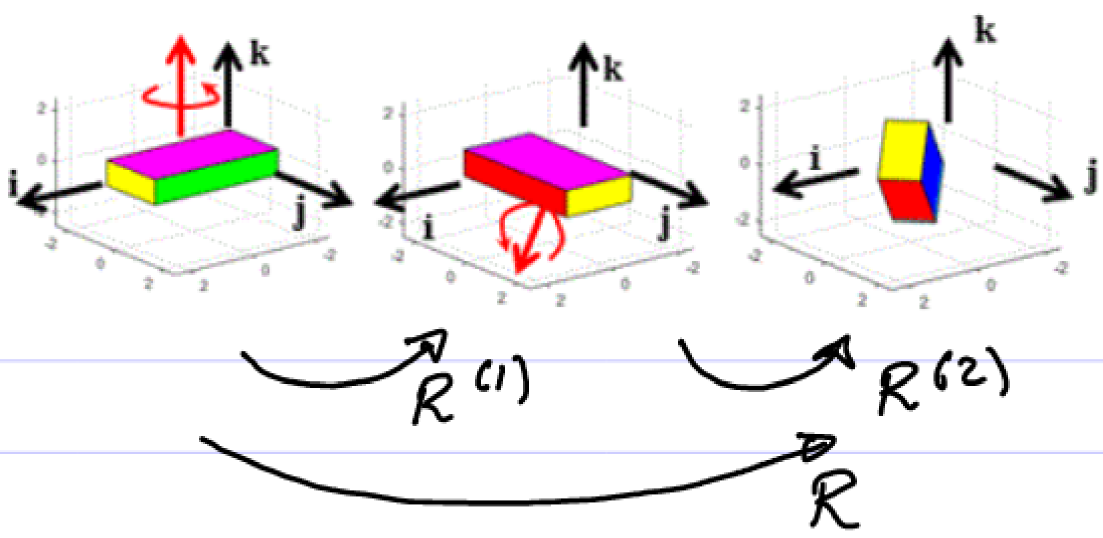
$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & \underbrace{(1-\cos\theta)n_y n_z - \sin\theta n_x}_{R_{yz}} \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & \underbrace{(1-\cos\theta)n_y n_z + \sin\theta n_x}_{R_{zy}} & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

Sum diagonals : $3\cos\theta + (1-\cos\theta)\overbrace{(n_x^2+n_y^2+n_z^2)}^{=1}$

$$R_{zy} - R_{yz} = 2\sin\theta\, n_x \quad \text{— etc ... for } \underline{n} \text{ formula}$$

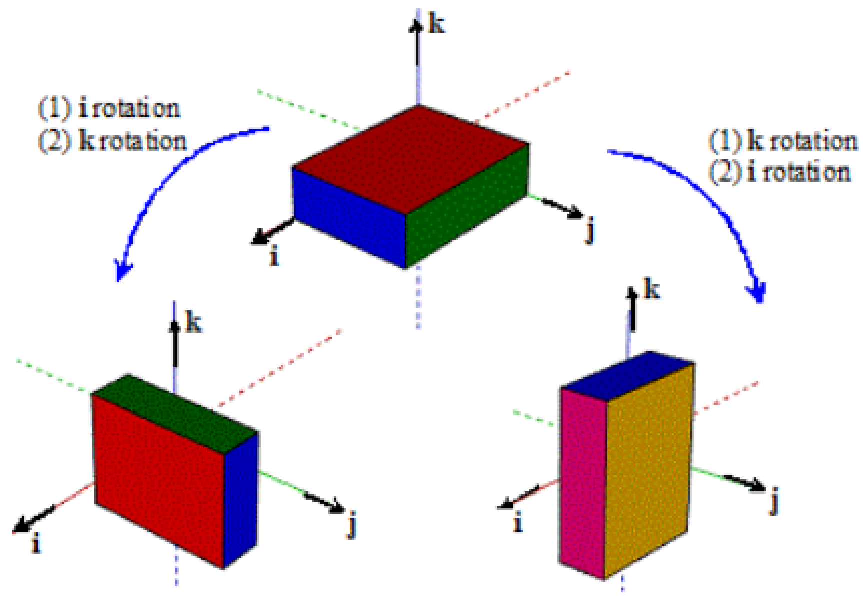
Sequence of rotations

$$R = R^{(2)} R^{(1)}$$



NB: order of rotations is important

$$R^{(2)} R^{(1)} \neq R^{(1)} R^{(2)}$$



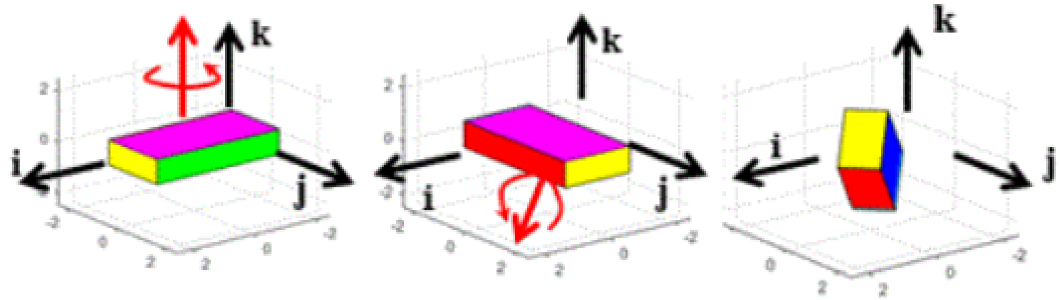
6.1.4 Example: An object is subjected to two successive rotations:

(1) A 90 degree rotation about the **k** axis (2) A 90 degree rotation about $\mathbf{n} = (\mathbf{i}+\mathbf{j})/\sqrt{2}$

(a) Find **R** for each rotation

(b) Find a single rotation from the initial to final orientation

(c) Find the axis and angle for the rotation in (b)



$$\mathbf{R} = \begin{bmatrix} \cos\theta + (1-\cos\theta)n_x^2 & (1-\cos\theta)n_x n_y - \sin\theta n_z & (1-\cos\theta)n_x n_z + \sin\theta n_y \\ (1-\cos\theta)n_x n_y + \sin\theta n_z & \cos\theta + (1-\cos\theta)n_y^2 & (1-\cos\theta)n_y n_z - \sin\theta n_x \\ (1-\cos\theta)n_x n_z - \sin\theta n_y & (1-\cos\theta)n_y n_z + \sin\theta n_x & \cos\theta + (1-\cos\theta)n_z^2 \end{bmatrix}$$

Rotation (1)

$\theta = \pi/2 \quad \underline{n} = \underline{k}$

$\Rightarrow n_x = n_y = 0 \quad n_z = 1$

$$\Rightarrow \boxed{R^{(1)} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

Rotation (2) $\quad \theta = \pi/2 \quad \underline{n} = (\underline{i}+\underline{j})/\sqrt{2} \quad n_x = n_y = 1/\sqrt{2} \quad n_z = 0$

$$\boxed{R^{(2)} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}}$$

(b) Formula $\quad R = R^{(2)} R^{(1)}$

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1/2 & -1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

(c) Formulas: $\quad 1 + 2\cos\theta = R_{xx} + R_{yy} + R_{zz} = 0$

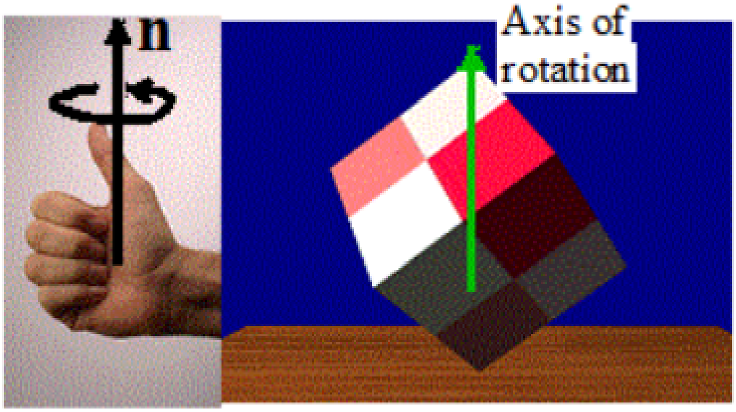
$$\Rightarrow \cos\theta = -1/2 \Rightarrow \boxed{\theta = 120^\circ}$$

$$\underline{\Omega} = \left(1/2\sin\theta\right)\left\{(R_{zy} - R_{yz})\underline{i} + (R_{xz} - R_{zx})\underline{j} + (R_{yx} - R_{xy})\underline{k}\right\}$$

$$= \frac{1}{2(\sqrt{3}/2)}\left\{\sqrt{2}\,\underline{i} + 0\,\underline{j} + \underline{k}\right\} = \boxed{(\sqrt{2}\,\underline{i} + \underline{k})/\sqrt{3}}$$

6.1.5 Angular Velocity and Acceleration

Angular velocity vector



$$\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$$

Magnitude: $|\underline{\omega}| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$

Rotation rate in rad/s

Direction $\underline{n} = \underline{\omega} / |\underline{\omega}|$ is parallel to axis of rotation (RH screw convention)

Angular acceleration

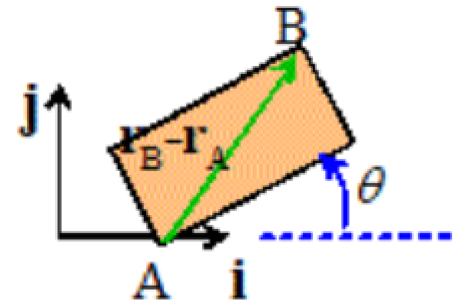
$$\underline{\alpha} = \alpha_x \underline{i} + \alpha_y \underline{j} + \alpha_z \underline{k} = d\underline{\omega}/dt$$

$$\alpha_x = \frac{d\omega_x}{dt} \qquad \alpha_y = \frac{d\omega_y}{dt} \qquad \alpha_z = \frac{d\omega_z}{dt}$$

Notes

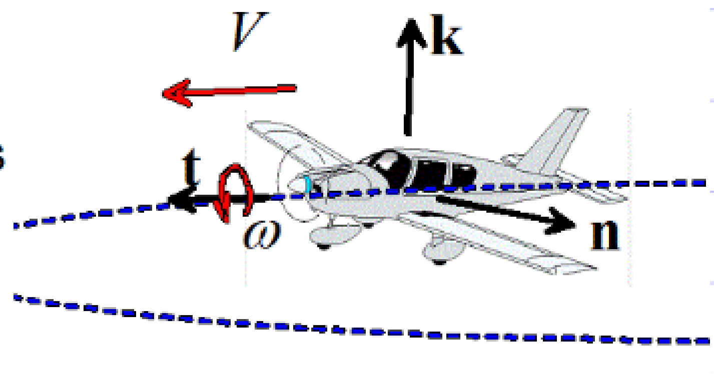
(1) Angular velocities add like vectors (see ex 6·1·6)

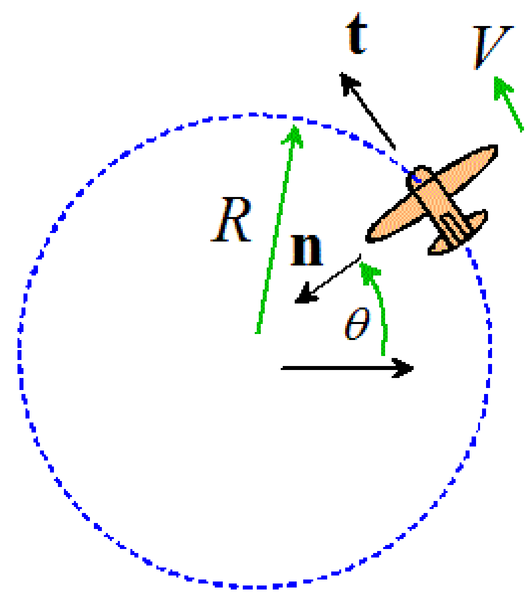
(2) $\underline{\omega}, \underline{\alpha}$ related by same calculus as $\underline{v}, \underline{a}$. Example: 2D rotation



	Rotation	Linear motion
Const accel:	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	$x = x_0 + v_0 t + \frac{1}{2} a t^2$
	$\omega = \omega_0 + \alpha t$	$v = v_0 + a t$
	$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$
General:	$\omega = d\theta/dt \quad \alpha = d\omega/dt$	$v = dx/dt \quad a = dv/dt$
	$\alpha = \omega \dfrac{d\omega}{d\theta}$	$a = v \dfrac{dv}{dx}$

6.1.6 Example: The propeller on the aircraft spins at 2000 rpm. The aircraft travels at 200 km/hr in a turn with radius 1 km. What is the angular velocity vector of
(i) the body of the aircraft, and
(ii) the propeller?





Body $\quad \underline{\omega}_{Body} = \frac{d\theta}{dt} \underline{k}$

Circular motion $\quad V = R\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{V}{R}$

$\Rightarrow \frac{d\theta}{dt} = \frac{200}{3600} = \frac{1}{18} \Rightarrow \boxed{\underline{\omega}_{Body} = \frac{1}{18} \underline{k}}$

Propeller

spins @ 2000 rpm about $\underline{t}$ relative to body

$$\Rightarrow \boxed{\underline{\omega}_{prop} = \underline{\omega}_{Body} + \frac{2000 \times 2\pi}{60} \underline{t} = \frac{1}{18} \underline{k} + \frac{200\pi}{3} \underline{t}}$$

6.1.7 Spin tensor & $\underline{\omega}$ - R relation

Let $\underline{\omega} = \omega_x \underline{i} + \omega_y \underline{j} + \omega_z \underline{k}$

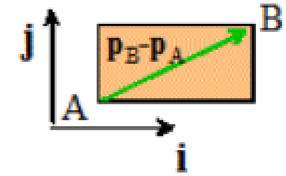
Define spin matrix (tensor)

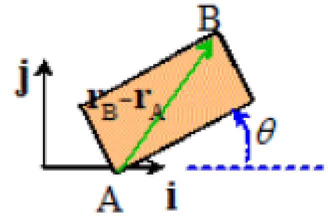
$$W = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \quad (3D)$$

$$W = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \quad (2D)$$

Then: $\boxed{W = \frac{dR}{dt} R^T}$ $\quad$ $\boxed{\frac{dR}{dt} = W R}$

6.1.8 Example: A rigid body rotates about the **k** axis as shown. Calculate the (2D) spin tensor and use it to determine the angular velocity vector.





$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow \frac{dR}{dt} = \frac{d\theta}{dt}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

Hence $$W = \frac{dR}{dt}R^T = \frac{d\theta}{dt}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

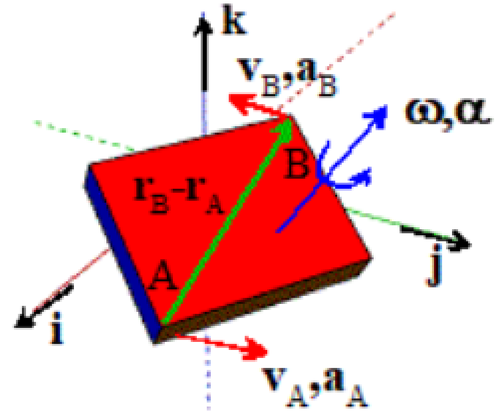
$$\Rightarrow \quad W = \frac{d\theta}{dt}\begin{bmatrix} 0 & -\sin^2\theta - \cos^2\theta \\ \sin^2\theta + \cos^2\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & -d\theta/dt \\ d\theta/dt & 0 \end{bmatrix}$$

Recall $$W = \begin{bmatrix} 0 & -\omega_z \\ \omega_z & 0 \end{bmatrix} \Rightarrow \boxed{\underset{\sim}{\omega} = \frac{d\theta}{dt}\underset{\sim}{k}}$$

6.1.9 Formulas for relative velocity and acceleration of two points in a rigid body

Given: $\underline{\omega}$, $\underline{\alpha}$, and $\underline{v}_A$, $\underline{a}_A$ for point A

Find $\underline{v}_B$, $\underline{a}_B$ for point B



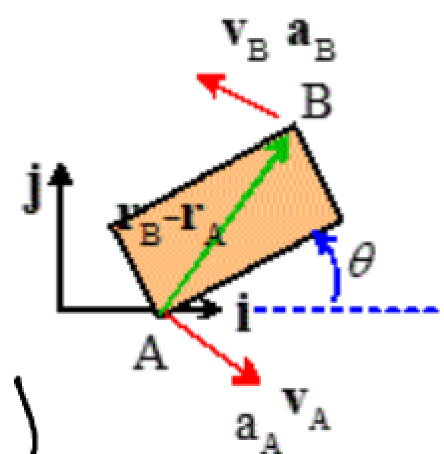
3D Formulas

Velocity: $\underline{v}_B - \underline{v}_A = \underline{\omega} \times (\underline{r}_B - \underline{r}_A)$

Accel: $\underline{a}_B - \underline{a}_A = \underline{\alpha} \times (\underline{r}_B - \underline{r}_A) + \underbrace{\underline{\omega} \times}_{\text{Second}} [\underbrace{\underline{\omega} \times (\underline{r}_B - \underline{r}_A)}_{\text{Do this first}}]$

2D Formulas

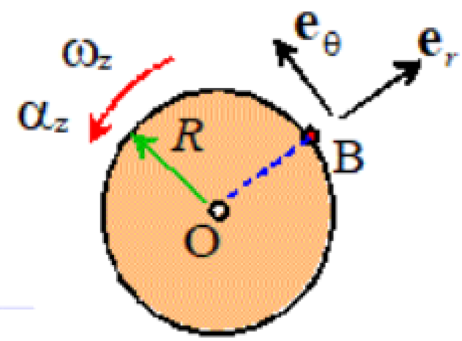
Note $\underline{\omega} = \omega_z \underline{k}$ $\quad \underline{\alpha} = \alpha_z \underline{k}$ $\quad \omega_z = \frac{d\theta}{dt}$



$$\underline{v}_B - \underline{v}_A = \omega_z \underline{k} \times (\underline{r}_B - \underline{r}_A)$$

$$\underline{a}_B - \underline{a}_A = \alpha_z \underline{k} \times (\underline{r}_B - \underline{r}_A) - \omega_z^2 (\underline{r}_B - \underline{r}_A)$$

6.1.10 Example: Point O on the disk is stationary. It spins with angular speed ω_z and angular acceleration α_z. Find the velocity and acceleration of point B



(O is stationary)

$\underline{\omega} = \omega_z \underline{k} \qquad \underline{\alpha} = \alpha_z \underline{k} \qquad$ Recall $\underline{k} \times \underline{e_r} = \underline{e_\theta}$

$\underline{k} \times \underline{e_\theta} = -\underline{e_r}$

Formulas: $\underline{v_B} - \underbrace{\cancel{\underline{v_0}}}_{=\underline{0}} = \omega_z \underline{k} \times (\underline{r_B} - \underline{r_0}) = \omega_z \underline{k} \times R \underline{e_r}$

$\Rightarrow \boxed{\underline{v_B} = \omega_z R \, \underline{e_\theta}}$

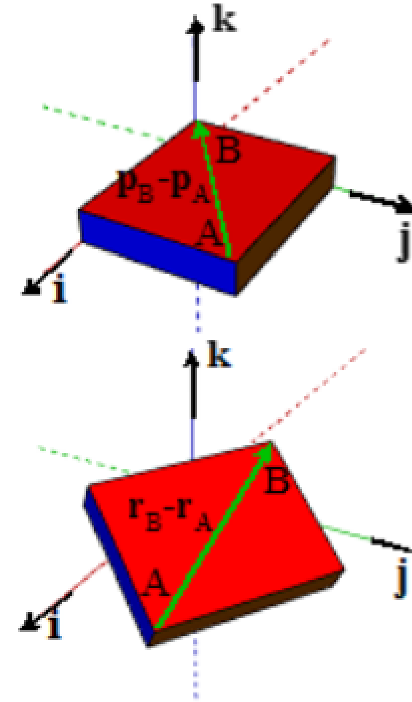
$\underline{a_B} - \underbrace{\cancel{\underline{a_0}}}_{=\underline{0}} = \alpha_z \underline{k} \times (\underline{r_B} - \underline{r_0}) - \omega_z^2 (\underline{r_B} - \underline{r_0})$

$= \alpha_z \underline{k} \times R \underline{e_r} - \omega_z^2 R \underline{e_r}$

$\boxed{\underline{a_B} = \alpha_z R \, \underline{e_\theta} - R \omega_z^2 \underline{e_r}}$

These are the circular motion formulas!

Proof of the rigid body kinematics formulas



(1) Prove that **R** is orthogonal $\mathbf{R}^T\mathbf{R} = \mathbf{1}$

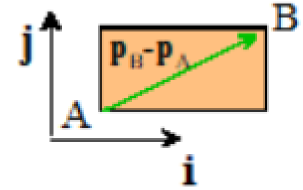
(2) Show $\left(\dfrac{d\mathbf{R}}{dt}\mathbf{R}^T\right)\mathbf{u} = \mathbf{W}\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u}$ for any vector $\mathbf{u}$

(3) Prove $\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega}\times(\mathbf{r}_B - \mathbf{r}_A)$

$$\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha}\times(\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega}\times\left[\boldsymbol{\omega}\times(\mathbf{r}_B - \mathbf{r}_A)\right]$$

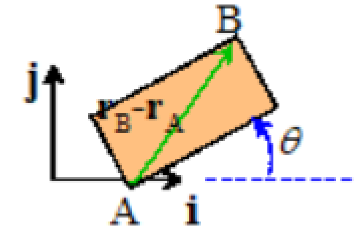
Proof that R is orthogonal

(1) Note length of AB is constant $|\mathbf{r}_B - \mathbf{r}_A| = |\mathbf{p}_B - \mathbf{p}_A|$



(2) Recall $(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$

(3) Hence $|\mathbf{r}_B - \mathbf{r}_A|^2 = (\mathbf{r}_B - \mathbf{r}_A)\cdot(\mathbf{r}_B - \mathbf{r}_A) = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)\cdot\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$



(4) Linear algebra formula $(\mathbf{A}\mathbf{a})\cdot(\mathbf{B}\mathbf{b}) = \mathbf{a}\cdot\left(\mathbf{A}^T\mathbf{B}\right)\mathbf{b}$

(5) Hence $\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)\cdot\mathbf{R}(\mathbf{p}_B - \mathbf{p}_A) = (\mathbf{p}_B - \mathbf{p}_A)\cdot\left(\mathbf{R}^T\mathbf{R}\right)(\mathbf{p}_B - \mathbf{p}_A)$

(6) Therefore, to preserve distance between A and B

$$|\mathbf{r}_B - \mathbf{r}_A|^2 = |\mathbf{p}_B - \mathbf{p}_A|^2$$

$$\Rightarrow (\mathbf{p}_B - \mathbf{p}_A)\cdot\left(\mathbf{R}^T\mathbf{R}\right)(\mathbf{p}_B - \mathbf{p}_A) = (\mathbf{p}_B - \mathbf{p}_A)\cdot(\mathbf{p}_B - \mathbf{p}_A)$$

$$\Rightarrow (\mathbf{p}_B - \mathbf{p}_A)\cdot\left(\mathbf{R}^T\mathbf{R} - \mathbf{1}\right)(\mathbf{p}_B - \mathbf{p}_A) = \mathbf{0}$$

$\Rightarrow \mathbf{R}^T\mathbf{R} = \mathbf{1}$ (because $\mathbf{u}\cdot\mathbf{A}\mathbf{u} = 0 \;\; \forall \mathbf{u} \;\; \Leftrightarrow \mathbf{A} = \mathbf{0}$ for a symmetric matrix)

Spin matrix - angular velocity relation

Prove $\left(\frac{d\mathbf{R}}{dt}\mathbf{R}^T\right)\mathbf{u} = \mathbf{W}\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u}$

(1) Note that $\mathbf{W}$ is skew, i.e. $\mathbf{W}^T = -\mathbf{W}$ Linear algebra formula $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Proof – recall $\mathbf{R}$ is orthogonal $\mathbf{R}\mathbf{R}^T = \mathbf{1}$

$$\Rightarrow \frac{d\mathbf{R}}{dt}\mathbf{R}^T + \mathbf{R}\frac{d\mathbf{R}^T}{dt} = \mathbf{0} \Rightarrow \mathbf{W} + \mathbf{W}^T = \mathbf{0}$$

(2) Skew matrices have property that $\mathbf{W}\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u}$ for all vectors $\mathbf{u}$

To see this note

$$\mathbf{W}\mathbf{u} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \omega_y u_z - \omega_z u_y \\ \omega_z u_x - \omega_x u_z \\ \omega_x u_y - \omega_y u_x \end{bmatrix}$$

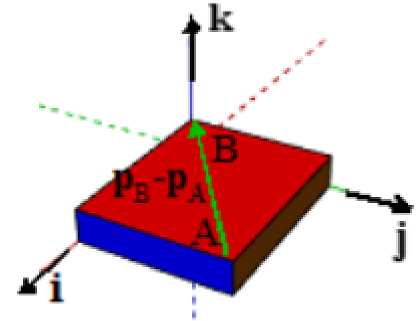
$$\boldsymbol{\omega}\times\mathbf{u} = (\omega_y u_z - \omega_z u_y)\mathbf{i} - (\omega_x u_z - \omega_z u_x)\mathbf{j} + (\omega_x u_y - \omega_y u_x)\mathbf{k}$$

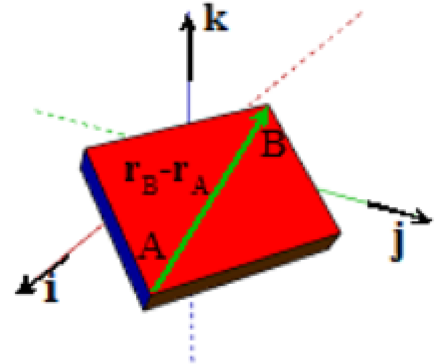
Angular velocity is therefore defined as follows:

(1) Define $\mathbf{W} = \frac{d\mathbf{R}}{dt}\mathbf{R}^T$

(2) Define $\boldsymbol{\omega}$ to be the vector satisfying $\mathbf{W}\mathbf{u} = \boldsymbol{\omega}\times\mathbf{u} \quad \forall\mathbf{u}$

Proof of the rigid body kinematics formulas





Prove that $\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$

By definition $\mathbf{r}_B - \mathbf{r}_A = \mathbf{R}(\mathbf{p}_B - \mathbf{p}_A)$

Differentiate $\mathbf{v}_B - \mathbf{v}_A = \dfrac{d}{dt}(\mathbf{r}_B - \mathbf{r}_A) = \dfrac{d\mathbf{R}}{dt}(\mathbf{p}_B - \mathbf{p}_A)$

Note that $\mathbf{R}^T(\mathbf{r}_B - \mathbf{r}_A) = (\mathbf{p}_B - \mathbf{p}_A)$

$$\Rightarrow \mathbf{v}_B - \mathbf{v}_A = \frac{d\mathbf{R}}{dt}\mathbf{R}^T(\mathbf{r}_B - \mathbf{r}_A)$$

Recall spin tensor $\dfrac{d\mathbf{R}}{dt}\mathbf{R}^T = \mathbf{W}$

Finally recall $\mathbf{W}\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}$ for any vector $\mathbf{u}$

$$\Rightarrow \mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)$$

Prove that $\mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)]$

Differentiate $\mathbf{a}_B - \mathbf{a}_A = \dfrac{d}{dt}(\mathbf{v}_B - \mathbf{v}_A) = \dfrac{d\boldsymbol{\omega}}{dt} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times \dfrac{d}{dt}(\mathbf{r}_B - \mathbf{r}_A)$

Recall $\dfrac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\alpha}$ $\quad \dfrac{d}{dt}(\mathbf{r}_B - \mathbf{r}_A) = \boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A) \quad \Rightarrow \quad \mathbf{a}_B - \mathbf{a}_A = \boldsymbol{\alpha} \times (\mathbf{r}_B - \mathbf{r}_A) + \boldsymbol{\omega} \times [\boldsymbol{\omega} \times (\mathbf{r}_B - \mathbf{r}_A)]$